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# A GAME THEORY APPROACH TO COOPERATIVE AND NON-COOPERATIVE ROUTING PROBLEMS

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**Abstract**

Previous work on multi-objective routing takes a system optimization approach to minimize some global objective function. In this paper, we take a different approach using a game theoretic formulation. We focus on a simple example of two classes which minimize a delay objective. We present three cases. The first case (baseline) does global optimization where the routing policies for the two classes are forced to be equal. The second case is where the two classes cooperate to minimize the same objective function of global average delay. In general, this team optimization approach will have a multiplicity of solutions which allow us to use secondary objectives to select the operating point. The third case is where each class optimizes its own objective function (which may or may not be identical)- this corresponds to the classical non-cooperative Nash game. This allows different objectives to be adopted by the different classes.

## 1. INTRODUCTION

The usual approach to distributed system design and control is the optimization of a single function, which may be the combination of multiple objectives as seen by the system administrator [2, 12]. Thus, it is assumed that all customers in the system cooperate for the socially optimum, such as optimizing the average customer performance.

However, in a real distributed-environment there is a diversity of customer classes, each one with possibly different objectives. These different classes of customers compete for the limited common resources of the distributed system in order to optimize their own objectives, ignoring the inconvenience that they cause to the other customer classes. For example, different telecommunication companies may share the same communication links and one of them may want to maximize the throughput of its customers, another may want to minimize its average customer delay and a third may want to minimize the blocking probability of its customers. Another example is when different users share a multiprocessor system and one group of users wants to maximize its throughput, similarly another group of users wants to maximize its own throughput, another group of users wants to minimize its average response time and finally another group of users wants to minimize the variance of its response time.

Customers of a given class arrive to the distributed system requiring transfer to a destination node. The problem of deciding through which path each customer will be routed is the routing problem. Kobayashi & Gerla [13] consider the single objective multiple class routing problem in closed queueing networks. Each closed chain corresponds to a different class of customers. They minimize the average delay, which is not convex, for closed chains routing, and therefore local minima exist. de Souza e Silva & Gerla [4] similarly consider the single objective load balancing problem in a product form queueing network with fixed closed chain routing. They minimize a measure of the average delay with respect to the open chains flows.

In this paper for simplicity of presentation, we consider two classes of customers which select between two links joining the entry point and the destination (an expanded version is [8] and the more general case is [9]). We formulate and solve the routing problem both as a team optimization problem and as a Nash non-cooperative game [1] among the two competing classes of customers, where each class of customers tries to operate in the most beneficial way for its own customers. The formulation of the routing problem as a Nash game has also been (independently) proposed by Bovopoulos [3]. Another optimization problem in distributed systems that has been recently formulated as a Nash game is the flow control problem [3, 5, 11]. We have also taken a different approach for distributed systems with priority classes. We have formulated and solved the two-priority classes load sharing problem as a Stackelberg game [7].

Other problems in distributed systems, where some resources are shared among competing classes of customers, may also be formulated as Nash or Stackelberg games. We have formulated and solved the join load sharing, routing and congestion control problem in arbitrary distributed systems with multiple competing classes as a Nash game [9], and a Stackelberg game [10].

## 2. NOTATION

Let class $k$ customers arrive to the system with rate $\lambda^k$ (Poisson arrivals). So, the total arrival rate is $\lambda = \sum_k \lambda^k$. Customers of any class may be served at any server, where server $i$ has rate $C_i$. So, the total system capacity is $C = \sum_i C_i$. Without loss of generality, let the service requirement of each customer be exponentially distributed with mean 1. The fraction of class $k$ customers assigned to server $i$ is $\phi_i^k$. Let also the superscript * at a variable denote the optimum value of that variable. Furthermore, for stability reasons it is assumed that the total arrival rate is less than the total service rate : $\lambda \leq C$.



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In this paper, for simplicity, we consider two classes and two servers, i.e. $k \in \{\alpha, \beta\}$, $i \in \{1, 2\}$. In the following sections, we consider three different formulations and solutions for sharing the two servers among customers of the two classes.

## 3. TRAFFIC AGGREGATION

In this section, we find the optimal routing policy, when the two classes are aggregated into a single class. Therefore, the fraction of class $\alpha$ customers assigned to a server is equal to the fraction of class $\beta$ customers assigned to that server, i.e. $\phi_1^\alpha = \phi_1^\beta = \phi_1$ and $\phi_2^\alpha = \phi_2^\beta = \phi_2$. If both classes want to minimize the average customer delay in the system [12], then we have the following optimization problem :

$$
\begin{aligned}
&\textit{minimize} \quad J(\phi_1, \phi_2) \;=\; \sum_{i=1}^{2} \frac{\phi_i}{C_i - (\lambda^\alpha + \lambda^\beta) * \phi_i} \\
&\textit{with respect to} \quad \phi_1, \ \phi_2 \\
&\textit{such that} \quad \phi_1 + \phi_2 = 1, \ \ \phi_1 \geq 0, \ \ \phi_2 \geq 0
\end{aligned}
$$

The average delay objective function $J(\phi_1, \phi_2)$ is convex with respect to $(\phi_1, \phi_2)$ over the convex space $\phi_1 + \phi_2 = 1$, $\phi_1 \geq 0$, $\phi_2 \geq 0$, for $C_1 - (\lambda^\alpha + \lambda^\beta) * \phi_1 > 0$ and $C_2 - (\lambda^\alpha + \lambda^\beta) * \phi_2 > 0$. This is a simple problem and can easily be solved [2, 6] :

$$
\begin{aligned}
&If \quad C_1 - \sqrt{C_1 C_2} \leq \lambda^\alpha - \lambda^\beta \ \ and \ \ C_2 - \sqrt{C_1 C_2} \leq \lambda^\alpha + \lambda^\beta \leq C_1 + C_2 \\
&\quad then \ \ \phi_1^* = \frac{C_1}{\lambda^\alpha - \lambda^\beta} - K_1, \quad \phi_2^* = 1 - \phi_1^* \\
&If \quad 0 \leq \lambda^\alpha + \lambda^\beta \leq C_1 - \sqrt{C_1 C_2}, \ \ then \ \ \phi_1^* = 1, \quad \phi_2^* = 0 \\
&If \quad 0 \leq \lambda^\alpha + \lambda^\beta \leq C_2 - \sqrt{C_1 C_2}, \ \ then \ \ \phi_1^* = 0, \quad \phi_2^* = 1
\end{aligned}
$$

where $K_1 = \dfrac{C_1 + C_2 - \lambda^\alpha - \lambda^\beta}{\lambda^\alpha + \lambda^\beta} * \dfrac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}}$.

## 4. TEAM OPTIMIZATION

In this section, we find the optimum routing decisions, when each class is treated independently from the other. The fraction of class $\alpha$ customers assigned to a server may be different than the fraction of class $\beta$ customers assigned to that server. However, both classes minimize the same objective - the average customer delay. This problem can be considered as a cooperative team game [1] between the two classes, where each class solves the following problem :

$$
\begin{aligned}
&\textit{minimize} \\
&J(\phi_1^\alpha, \phi_2^\alpha, \phi_1^\beta, \phi_2^\beta) \;=\; \sum_{i=1}^{2} \frac{\lambda^\alpha * \phi_i^\alpha + \lambda^\beta * \phi_i^\beta}{\lambda^\alpha + \lambda^\beta} * \frac{1}{C_i - \lambda^\alpha * \phi_i^\alpha - \lambda^\beta * \phi_i^\beta} \\
&\textit{with respect to} \quad \phi_1^\alpha, \ \phi_2^\alpha, \ \phi_1^\beta, \ \phi_2^\beta \\
&\textit{such that} \quad \phi_1^\alpha + \phi_2^\alpha = 1, \ \ \phi_1^\beta + \phi_2^\beta = 1, \\
&\qquad\qquad \phi_1^\alpha, \ \phi_2^\alpha, \ \phi_1^\beta, \ \phi_2^\beta \ \geq 0
\end{aligned}
$$

The objective function $J(\phi_1^\alpha, \phi_2^\alpha, \phi_1^\beta, \phi_2^\beta)$ is convex with respect to $(\phi_1^\alpha, \phi_2^\alpha, \phi_1^\beta, \phi_2^\beta)$ over the convex space $\phi_1^\alpha + \phi_2^\alpha = 1$, $\phi_1^\beta + \phi_2^\beta = 1$, $\phi_1^\alpha, \phi_2^\alpha, \phi_1^\beta, \phi_2^\beta \geq 0$, for $C_1 - \lambda^\alpha * \phi_1^\alpha - \lambda^\beta * \phi_1^\beta > 0$ and $C_2 - \lambda^\alpha * \phi_2^\alpha - \lambda^\beta * \phi_2^\beta > 0$.

Define the auxiliary variables

$$
\begin{aligned}
K_1^\alpha &= \frac{C_1 + C_2 - \lambda^\alpha - \lambda^\beta}{\lambda^\alpha} * \frac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}} \\
K_1^\beta &= \frac{C_1 + C_2 - \lambda^\alpha - \lambda^\beta}{\lambda^\beta} * \frac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}}
\end{aligned}
$$

Then, the following policy [8] will optimally assign the arriving customers to the two servers:

$$
\begin{aligned}
&If \quad \lambda^\alpha + \lambda^\beta \leq C_1 + C_2, \\
&\quad then \quad \phi_1^{\alpha*} = \frac{C_1 - \lambda^\beta \phi_1^{\beta*}}{\lambda^\alpha} - K_1^\alpha \\
&\qquad\qquad \phi_1^{\beta*} = \frac{C_1 - \lambda^\alpha \phi_1^{\alpha*}}{\lambda^\beta} - K_1^\beta \\
&\quad accept \ \ the \ \ solution \ \ only \ \ if \\
&\qquad C_1 - \lambda^\beta \phi_1^{\beta*} - \sqrt{\frac{C_1}{C_2}}(C_2 - \lambda^\beta \phi_2^{\beta*}) \leq \lambda^\alpha \\
&\qquad C_2 - \lambda^\beta \phi_2^{\beta*} - \sqrt{\frac{C_2}{C_1}}(C_1 - \lambda^\beta \phi_1^{\beta*}) \leq \lambda^\alpha \\
&\qquad C_1 - \lambda^\alpha \phi_1^{\alpha*} - \sqrt{\frac{C_1}{C_2}}(C_2 - \lambda^\alpha \phi_2^{\alpha*}) \leq \lambda^\beta \\
&\qquad C_2 - \lambda^\alpha \phi_2^{\alpha*} - \sqrt{\frac{C_2}{C_1}}(C_1 - \lambda^\alpha \phi_1^{\alpha*}) \leq \lambda^\beta \\
&If \quad \lambda^\alpha + \lambda^\beta \leq C_1 - \sqrt{C_1 C_2}, \ \ then \ \ \phi_1^{\alpha*} = 1, \quad \phi_1^{\beta*} = 1 \\
&If \quad \lambda^\alpha \sqrt{C_2} - \lambda^\beta \sqrt{C_1} = \sqrt{C_1 C_2}(\sqrt{C_1} - \sqrt{C_2}), \\
&\quad then \ \ \phi_1^{\alpha*} = 1, \quad \phi_1^{\beta*} = 0 \\
&If \quad \lambda^\alpha \sqrt{C_1} - \lambda^\beta \sqrt{C_2} = \sqrt{C_1 C_2}(\sqrt{C_2} - \sqrt{C_1}), \\
&\quad then \ \ \phi_1^{\alpha*} = 0, \quad \phi_1^{\beta*} = 1 \\
&If \quad \lambda^\alpha + \lambda^\beta \leq C_2 - \sqrt{C_1 C_2}, \ \ then \ \ \phi_1^{\alpha*} = 0, \quad \phi_1^{\beta*} = 0 \\
&If \quad \lambda^\alpha + \lambda^\beta \geq C_1 - \sqrt{C_1 C_2} \ \ and \\
&\quad \lambda^\alpha \sqrt{C_2} - \lambda^\beta \sqrt{C_1} \leq \sqrt{C_1 C_2}(\sqrt{C_1} - \sqrt{C_2}), \\
&\quad then \quad \phi_1^{\alpha*} = 1, \quad \phi_1^{\beta*} = \frac{C_1 - \lambda^\alpha}{\lambda^\beta} - K_1^\beta \\
&If \quad \lambda^\alpha + \lambda^\beta \geq C_2 - \sqrt{C_1 C_2} \ \ and \\
&\quad \lambda^\alpha \sqrt{C_1} - \lambda^\beta \sqrt{C_2} \leq \sqrt{C_1 C_2}(\sqrt{C_2} - \sqrt{C_1}), \\
&\quad then \quad \phi_1^{\alpha*} = 0, \quad \phi_1^{\beta*} = \frac{C_1}{\lambda^\beta} - K_1^\beta \\
&If \quad \lambda^\alpha + \lambda^\beta \geq C_1 - \sqrt{C_1 C_2} \ \ and \\
&\quad \lambda^\beta \sqrt{C_2} - \lambda^\alpha \sqrt{C_1} \leq \sqrt{C_1 C_2}(\sqrt{C_1} - \sqrt{C_2}), \\
&\quad then \quad \phi_1^{\beta*} = 1, \quad \phi_1^{\alpha*} = \frac{C_1 - \lambda^\beta}{\lambda^\alpha} - K_1^\alpha \\
&If \quad \lambda^\alpha + \lambda^\beta \geq C_2 - \sqrt{C_1 C_2} \ \ and \\
&\quad \lambda^\beta \sqrt{C_1} - \lambda^\alpha \sqrt{C_2} \leq \sqrt{C_1 C_2}(\sqrt{C_2} - \sqrt{C_1}), \\
&\quad then \quad \phi_1^{\beta*} = 0, \quad \phi_1^{\alpha*} = \frac{C_1}{\lambda^\alpha} - K_1^\alpha
\end{aligned}
$$

Of course, the optimum routing fractions to the other server are $\phi_2^{\alpha*} = 1 - \phi_1^{\alpha*}$ and $\phi_2^{\beta*} = 1 - \phi_1^{\beta*}$.

In the first case, we choose $\phi_1^{\alpha*}$ which leads to a value for $\phi_1^{\beta*}$. The choise of value for $\phi_1^{\alpha*}$ is arbitrary so we may use some other criterion to decide which values to use.

In Fig. 1, we show the optimum routing fractions $(\phi_1^{\alpha*}, \phi_1^{\beta*})$ for fixed server capacities, $C_1 = 2$, $C_2 = 1$, fixed class $\beta$ arrival rate, $\lambda^\beta = 1$, and different class $\alpha$ arrival rates, $\lambda^\alpha = 0.1, ..., 1.9$. We notice something remarkable. The straight line solutions for different class $\alpha$ arrival rates intersect at a single intersection point. This means that there is a common pair of optimum routing fractions $(\phi_1^{\alpha*}, \phi_1^{\beta*})$, where we can optimally operate for different class $\alpha$ arrival rates. So, we can use the optimum routing fractions of the intersection point and operate optimally even if the class $\alpha$ arrival rate varies. Proposition  describes this result more formally.

**Proposition** : *Let two classes of customers $\alpha$ and $\beta$ cooperate in sharing two servers. Customers from each class arrive according to Poisson distribution and require service according to exponential distribution. Both classes minimize the average customer delay.*

***For a given system $C_1 > C_2$, with fixed class $\beta$ arrival rate $\lambda^\beta$, if $0 \le C_1 - (C_1 + C_2 - \lambda^\beta) * \frac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}} \le \lambda^\beta$,***

*then the straight lines of the team optimum fractions $(\phi_1^{\alpha *}, \phi_1^{\beta *})$, for different class $\alpha$ arrival rates $\lambda^\alpha$ ( $\lambda^\alpha + \lambda^\beta \le C_1 + C_2$), intersect at a single point*

$$(\phi_1^{\alpha *}, \phi_1^{\beta *}) = \left(\frac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}}, \frac{C_1}{\lambda^\beta} - \frac{C_1 + C_2 - \lambda^\beta}{\lambda^\beta} * \frac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}}\right)$$

*i.e. this intersection point is independent of the class $\alpha$ arrival rate.*

As we have seen we have a set of optimum routing fraction pairs $(\phi_1^{\alpha *}, \phi_1^{\beta *})$ that all achieve the same global minimum delay. However, these optimum routing fractions will give different average delays for each class. So, we can choose the operating point using another delay objective. In Fig. 2, we show the difference in the average delay of class $\alpha$ and class $\beta$ customers, $J^{\alpha *} - J^{\beta *}$, versus the class $\alpha$ optimum routing fraction, $\phi_1^{\alpha *}$, for fixed server capacities, $C_1 = 2$, $C_2 = 1$, fixed class $\beta$ arrival rate, $\lambda^\beta = 1$, and different class $\alpha$ arrival rates, $\lambda^\alpha = 0.1, ..., 1.9$. An example is when it is desired that both classes have the same average delay. Then this point will be the intersection of the delay difference line and the zero delay difference line. The operating point for this case is the same as the solution of section 3, where we aggregate the two classes into a single class and therefore we treat them similarly. Another example is when there is a secondary objective that class $\alpha$ should receive better treatment than class $\beta$. Then the lowest point of the delay difference line $J^{\alpha *} - J^{\beta *}$ is chosen.

## 5. NASH EQUILIBRIUM

In this section, we find the optimum routing decisions, when each class chooses the best strategy for its customers given the decision of the other class. Class $\alpha$ assigns its customers to the two servers such that the average delay of its customers is minimized. Similarly, class $\beta$ assigns its customers to the two servers such that the average delay of its customers is minimized. Therefore customers of different classes do not have the same objective and they compete for sharing the two servers. We formulate and solve the above multiobjective optimization problem as a non-cooperative Nash game [1] between the two classes. After reaching a Nash equilibrium, no class of customers will have a rational motive to unilaterally deviate from its equilibrium strategy.

Class $\alpha$ solves the following problem :

$$\begin{aligned} & \textit{minimize} \\ & J^\alpha(\phi_1^\alpha, \phi_2^\alpha, \phi_1^{\beta *}, \phi_2^{\beta *}) \quad = \quad \sum_{i=1}^{2} \frac{\phi_i^\alpha}{C_i - \lambda^\alpha * \phi_i^\alpha - \lambda^\beta * \phi_i^{\beta *}} \\ & \textit{with respect to} \qquad \phi_1^\alpha, \ \phi_2^\alpha \\ & \textit{such that} \qquad \phi_1^\alpha + \phi_2^\alpha = 1, \quad \phi_1^\alpha, \ \phi_2^\alpha \ \ge 0 \end{aligned}$$

The objective function $J^\alpha(\phi_1^\alpha, \phi_2^\alpha, \phi_1^{\beta *}, \phi_2^{\beta *})$ is convex with respect to $(\phi_1^\alpha, \phi_2^\alpha)$ over the convex space $\phi_1^\alpha + \phi_2^\alpha = 1$, $\phi_1^\alpha$, $\phi_2^\alpha \ge 0$, for $C_1 - \lambda^\alpha * \phi_1^\alpha - \lambda^\beta * \phi_1^\beta > 0$ and $C_2 - \lambda^\alpha * \phi_2^\alpha - \lambda^\beta * \phi_2^\beta > 0$.

**Class $\beta$ solves a similar problem using the optimal value for $(\phi_1^{\alpha *}, \phi_2^{\alpha *})$.**

**When the players are in a Nash equilibrium, no player can improve his cost by altering his decision unilaterally.** Next, we give the definition of a Nash equilibrium [1] in our context:

**Definition** : *A vector $[\phi_1^\alpha, \phi_2^\alpha, \phi_1^\beta, \phi_2^\beta]$ with $\phi_1^\alpha + \phi_2^\alpha = 1$, $\phi_1^\beta + \phi_2^\beta = 1$, and $\phi_1^\alpha$, $\phi_2^\alpha$, $\phi_1^\beta$, $\phi_2^\beta \ge 0$ is called a Nash equilibrium for a two-class routing game iff*

$$J^\alpha(\phi_1^{\alpha *}, \phi_2^{\alpha *}, \phi_1^{\beta *}, \phi_2^{\beta *}) \quad \le \inf_{\substack{\phi_1^\alpha + \phi_2^\alpha = 1, \\ \phi_1^\alpha, \phi_2^\alpha \ge 0}} J^\alpha(\phi_1^\alpha, \phi_2^\alpha, \phi_1^{\beta *}, \phi_2^{\beta *})$$

$$J^\beta(\phi_1^{\alpha *}, \phi_2^{\alpha *}, \phi_1^{\beta *}, \phi_2^{\beta *}) \quad \le \inf_{\substack{\phi_1^\beta + \phi_2^\beta = 1, \\ \phi_1^\beta, \phi_2^\beta \ge 0}} J^\beta(\phi_1^{\alpha *}, \phi_2^{\alpha *}, \phi_1^\beta, \phi_2^\beta)$$

Therefore each class minimizes its average customer delay given that the other class has minimized the average delay of its customers.

Proof of existence and uniqueness of a solution can be found in our report [8]. Next, we find this unique Nash equilibrium for the above routing game.

Define the auxiliary variables

$$N_1^{\alpha *}(\phi_1^{\beta *}) \quad = \quad \frac{C_1 + C_2 - \lambda^\alpha - \lambda^\beta}{\lambda^\alpha} * \frac{\sqrt{C_1 - \lambda^\beta * \phi_1^{\beta *}}}{\sqrt{C_1 - \lambda^\beta * \phi_1^{\beta *}} + \sqrt{C_2 - \lambda^\beta * \phi}}$$

$$N_1^{\beta *}(\phi_1^{\alpha *}) \quad = \quad \frac{C_1 + C_2 - \lambda^\alpha - \lambda^\beta}{\lambda^\beta} * \frac{\sqrt{C_1 - \lambda^\alpha * \phi_1^{\alpha *}}}{\sqrt{C_1 - \lambda^\alpha * \phi_1^{\alpha *}} + \sqrt{C_2 - \lambda^\alpha * \phi}}$$

Then, the following policy [8] will route the arriving customers to the two servers such that a Nash equilibrium is achieved:

$$\begin{aligned}
& \textit{If} \quad \lambda^\alpha + \lambda^\beta \le C_1 + C_2, \\
& \quad \textit{then} \quad \phi_1^{\alpha *} = \frac{C_1 - \lambda^\beta \phi_1^{\beta *}}{\lambda^\alpha} - N_1^\alpha(\phi_1^{\beta *}) \\
& \qquad\qquad \phi_1^{\beta *} = \frac{C_1 - \lambda^\alpha \phi_1^{\alpha *}}{\lambda^\beta} - N_1^\beta(\phi_1^{\alpha *}) \\
& \quad \textit{accept the solution only if} \\
& \qquad\qquad C_1 - \lambda^\beta \phi_1^{\beta *} - \sqrt{(C_1 - \lambda^\beta \phi_1^{\beta *})(C_2 - \lambda^\beta \phi_2^{\beta *})} \le \lambda^\alpha \\
& \qquad\qquad C_2 - \lambda^\beta \phi_2^{\beta *} - \sqrt{(C_1 - \lambda^\beta \phi_1^{\beta *})(C_2 - \lambda^\beta \phi_2^{\beta *})} \le \lambda^\alpha \\
& \qquad\qquad C_1 - \lambda^\alpha \phi_1^{\alpha *} - \sqrt{(C_1 - \lambda^\alpha \phi_1^{\alpha *})(C_2 - \lambda^\alpha \phi_2^{\alpha *})} \le \lambda^\beta \\
& \qquad\qquad C_2 - \lambda^\alpha \phi_2^{\alpha *} - \sqrt{(C_1 - \lambda^\alpha \phi_1^{\alpha *})(C_2 - \lambda^\alpha \phi_2^{\alpha *})} \le \lambda^\beta \\
& \textit{If} \quad \lambda^\alpha + \lambda^\beta \le C_1 - \sqrt{(C_1 - \lambda^\alpha) C_2} \ \textit{and} \\
& \quad \lambda^\alpha + \lambda^\beta \le C_1 - \sqrt{(C_1 - \lambda^\beta) C_2}, \\
& \quad \textit{then} \ \phi_1^{\alpha *} = 1, \quad \phi_1^{\beta *} = 1 \\
& \textit{If} \quad 0 \le \lambda^\alpha \le C_1 - \sqrt{(C_1 (C_2 - \lambda^\beta)} \ \textit{and} \\
& \quad 0 \le \lambda^\beta \le C_2 - \sqrt{(C_1 - \lambda^\alpha) C_2}, \\
& \quad \textit{then} \ \phi_1^{\alpha *} = 1, \quad \phi_1^{\beta *} = 0 \\
& \textit{If} \quad 0 \le \lambda^\alpha \le C_2 - \sqrt{(C_1 - \lambda^\beta) C_2} \ \textit{and} \\
& \quad 0 \le \lambda^\beta \le C_1 - \sqrt{C_1 (C_2 - \lambda^\alpha)}, \\
& \quad \textit{then} \ \phi_1^{\alpha *} = 0, \quad \phi_1^{\beta *} = 1 \\
& \textit{If} \quad \lambda^\alpha + \lambda^\beta \le C_2 - \sqrt{C_1 (C_2 - \lambda^\alpha)} \ \textit{and} \\
& \quad \lambda^\alpha + \lambda^\beta \le C_2 - \sqrt{C_1 (C_2 - \lambda^\beta)}, \\
& \quad \textit{then} \ \phi_1^{\alpha *} = 0, \quad \phi_1^{\beta *} = 0 \\
& \textit{If} \quad \lambda^\alpha + \lambda^\beta \ge C_1 - \sqrt{(C_1 - \lambda^\alpha) C_2} \ \textit{and} \\
& \quad \lambda^\beta \ge C_2 - \sqrt{(C_1 - \lambda^\alpha) C_2}, \\
& \quad \textit{then} \quad \phi_1^{\alpha *} = 1, \quad \phi_1^{\beta *} = \frac{C_1 - \lambda^\alpha}{\lambda^\beta} - N_1^\beta(1) \\
& \quad \textit{accept the solution only if} \\
& \qquad\qquad \lambda^\alpha \le C_1 - \lambda^\beta \phi_1^{\beta *} - \sqrt{(C_1 - \lambda^\beta \phi_1^{\beta *})(C_2 - \lambda^\beta \phi_2^{\beta *})}
\end{aligned}$$

$$
\begin{aligned}
&If \quad \lambda^\alpha + \lambda^\beta \geq C_2 - \sqrt{C_1(C_2 - \lambda^\alpha)} \quad and \\
&\quad \lambda^\beta \geq C_1 - \sqrt{C_1(C_2 - \lambda^\alpha)}, \\
&\quad then \quad \phi_1^{\alpha *} = 0, \quad \phi_1^{\beta *} = \frac{C_1}{\lambda^\beta} - N_1^\beta(0) \\
&\quad accept \quad the \ solution \ only \ if \\
&\qquad \lambda^\alpha \leq C_2 - \lambda^\beta \phi_2^{\beta *} - \sqrt{(C_1 - \lambda^\beta \phi_1^{\beta *})(C_2 - \lambda^\beta \phi_2^{\beta *})} \\
&If \quad \lambda^\alpha + \lambda^\beta \geq C_1 - \sqrt{(C_1 - \lambda^\beta)C_2} \quad and \\
&\quad \lambda^\alpha \geq C_2 - \sqrt{(C_1 - \lambda^\beta)C_2}, \\
&\quad then \quad \phi_1^{\beta *} = 1, \quad \phi_1^{\alpha *} = \frac{C_1 - \lambda^\beta}{\lambda^\alpha} - N_1^\alpha(1) \\
&\quad accept \quad the \ solution \ only \ if \\
&\qquad \lambda^\beta \leq C_1 - \lambda^\alpha \phi_1^{\alpha *} - \sqrt{(C_1 - \lambda^\alpha \phi_1^{\alpha *})(C_2 - \lambda^\alpha \phi_2^{\alpha *})} \\
&If \quad \lambda^\alpha + \lambda^\beta \geq C_2 - \sqrt{C_1(C_2 - \lambda^\beta)} \quad and \\
&\quad \lambda^\alpha \geq C_1 - \sqrt{C_1(C_2 - \lambda^\beta)}, \\
&\quad then \quad \phi_1^{\beta *} = 0, \quad \phi_1^{\alpha *} = \frac{C_1}{\lambda^\alpha} - N_1^\alpha(0) \\
&\quad accept \quad the \ solution \ only \ if \\
&\qquad \lambda^\beta \leq C_2 - \lambda^\alpha \phi_2^{\alpha *} - \sqrt{(C_1 - \lambda^\alpha \phi_1^{\alpha *})(C_2 - \lambda^\alpha \phi_2^{\alpha *})}
\end{aligned}
$$

Of course, the Nash equilibrium routing fractions to the other server are $\phi_2^{\alpha *} = 1 - \phi_1^{\alpha *}$ and $\phi_2^{\beta *} = 1 - \phi_1^{\beta *}$.

In order to find the Nash equilibrium routing fractions $(\phi_1^{\alpha *}, \phi_1^{\beta *})$ for the first case of the Nash routing, we use a simultaneous adjustment algorithm. So, starting with $\phi_1^{\alpha *}(0) = \phi_1^{\beta *}(0) = 0$, we iterate according to the following algorithm:

$$\phi_1^\alpha(k+1) = \frac{C_1 - \lambda^\beta \phi_1^\beta(k)}{\lambda^\alpha} - N_1^\alpha(\phi_1^\beta(k))$$

$$\phi_1^\beta(k+1) = \frac{C_1 - \lambda^\alpha \phi_1^\alpha(k)}{\lambda^\beta} - N_1^\beta(\phi_1^\alpha(k))$$

In Fig. 3, we show the average delay difference between the two classes $J^{\alpha *} - J^{\beta *}$ for fixed server capacities $C_1 = 2, C_2 = 1$, fixed class $\beta$ arrival rate $\lambda^\beta = 1$ and different class $\alpha$ arrival rates $\lambda^\alpha$. When the class $\alpha$ arrival rate is equal to the class $\beta$ arrival rate $\lambda^\alpha = \lambda^\beta = 1$, then both classes have the same average delay. When a class has larger arrival rate then it also has larger average delay. For a very small class $\alpha$ arrival rate $\lambda^\alpha$, we notice something peculiar: the average delay difference curve is not monotonic with the arrival rate. This happens because for these values we hit the boundary ($\phi_1^{\alpha *} = 1$), as we see in Fig. 8.

In Fig. 4, we show the Nash equilibrium routing fractions of the two classes $\phi_1^{\alpha *}$ and $\phi_1^{\beta *}$, for fixed server capacities, $C_1 = 2, C_2 = 1$, fixed class $\beta$ arrival rate, $\lambda^\beta = 1$ and different class $\alpha$ arrival rates, $\lambda^\alpha$. We see that for very small class $\alpha$ arrival rate $\lambda^\alpha$, class $\alpha$ uses exclusively the faster server 1 ($\phi_1^{\alpha *} = 1$). For equal arrival rates $\lambda^\alpha = \lambda^\beta = 1$, the Nash equilibrium routing fractions intersect at the point $\phi_1^{\alpha *} = \phi_1^{\beta *}$. As we increase the arrival rate they depart each other to meet again when the arrival rate becomes large.

## 5. CONCLUSIONS

In this paper, we *formulate* and *solve* a *two class routing* problem. When the two classes of customers cooperate to minimize the average customer delay, then we formulate and solve the problem as a *team optimization problem*. When the two classes of customers compete among themselves and each class wants to minimize the average delay of its own customers, we introduce an alternative methodology for multiobjective performance optimization. In this case, we formulate and solve the problem as a *non-cooperative Nash game*. Each class of customers chooses the best strategy for its customers. A Nash equilibrium is achieved, where no class of customers has a rational motive to unilaterally depart from its strategy.

In summary, we have presented a novel approach which leads itself to multi-objective optimization problems.

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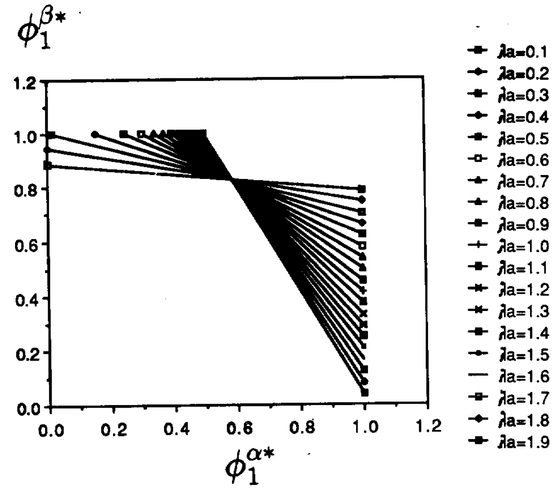


**Fig. 1** The optimum routing probabilities $(\phi_1^{\alpha*}, \phi_1^{\beta*})$ for fixed server capacities $C_1 = 2$ and $C_2 = 1$, fixed class $\beta$ arrival rate $\lambda^\beta = 1$ and different class $\alpha$ arrival rates $\lambda^\alpha = 0.1, ..., 1.9$.

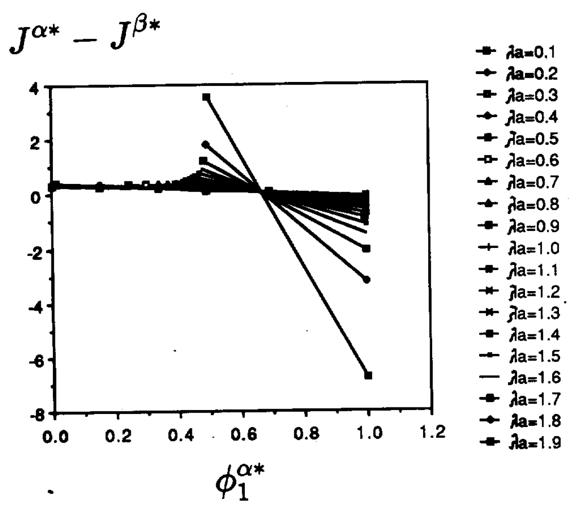


**Fig. 2** The difference of the optimum average delay of class $\alpha$ minus the optimum average delay of class $\beta$, $J^{\alpha*} - J^{\beta*}$, for fixed server capacities $C_1 = 2$ and $C_1 = 1$, fixed class $\beta$ arrival rate $\lambda^\beta = 1$ and different class $\alpha$ arrival rates $\lambda^\alpha = 0.1, ..., 1.9$.

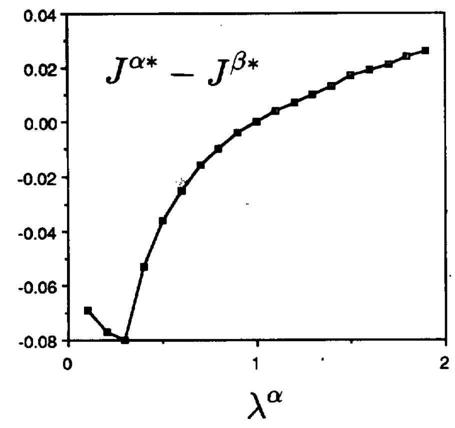


Fig. 3 The difference of the Nash equilibrium average delays of class $\alpha$ and class $\beta$, $J^{\alpha*} - J^{\beta*}$, for fixed server capacities $C_1 = 2$ and $C_2 = 1$, fixed class $\beta$ arrival rate $\lambda^\beta = 1$ and different class $\alpha$ arrival rates.

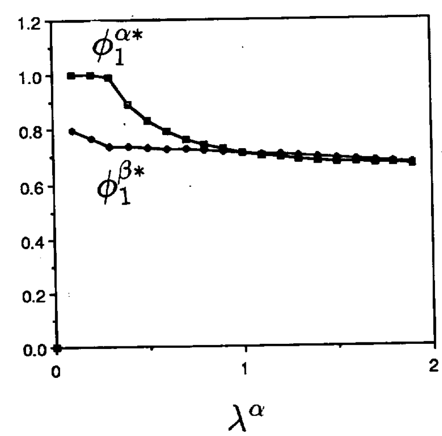


Fig. 4 The Nash equilibrium routing probabilities of class $\alpha$, $\phi_1^{\alpha*}$ and class $\beta$, $\phi_1^{\beta*}$, for fixed server capacities $C_1 = 2$ and $C_2 = 1$, fixed class $\beta$ arrival rate $\lambda^\beta = 1$ and different class $\alpha$ arrival rates $\lambda^\alpha$.